Indian Statistical Institute, Bangalore

B. Math.

First Year, First Semester

Analysis I

Final examination

Total Marks: 110 Maximum marks: 100

Date : Nov. 6, 2013

Time: 3 hours

1. Suppose $\{a_n\}_{n\geq 1}$ is a bounded sequence and $\{b_n\}_{n\geq 1}$ is a sequence converging to 0 as n tends to ∞. Show that $\{a_nb_n\}_{n\geq 1}$ converges to 0 as n tends to ∞. [10]

2. Let $\{x_n\}_{n\geq 1}$ be a bounded sequence of real numbers and $c = \liminf_{n\to\infty} x_n$. Show that for any $\epsilon > 0$ the set $M_\epsilon = \{n : x_n < c - \epsilon\}$ is a finite set. [10]

3. Show that the polynomial $p(x) = 3x^5 + 2x^3 + 6x + 5$ has exactly one real root. [10]

4. Let X be the set of all finite subsets of $\mathbb{N}$. Show that X is countable. [20]

5. State and prove the mean value theorem (you may assume Rolle's theorem). [20]

6. Let $f, g : [0,1] \to \mathbb{R}$ be continuous functions. Define h, k on $[0,1]$ by $h(x) =\text{Min}\{f(x), g(x)\}$ and $k(x) = \text{Max}\{f(x), g(x)\}$. Show that h, k are continuous. Give examples to show that both h, k need not be differentiable, even if f, g are differentiable. [20]

7. Suppose $a : \mathbb{R} \to \mathbb{R}$ is a function such that

$$a(\frac{x+y}{2}) \leq \frac{a(x)+a(y)}{2} \qquad x, y \in \mathbb{R}.$$

(i) Show that for all $n \geq 2$ and for all $x_1, x_2, \ldots, x_n$ in $\mathbb{R}$,

$$a\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leq \frac{1}{n}\sum_{i=1}^{n} a(x_i).$$

(ii) If a is continuous show that

$$a(px + (1-p)y) \leq pa(x) + (1-p)a(y)$$

for all $x, y \in \mathbb{R}$ and $0 \leq p \leq 1$. (Hint: First prove the result for rational numbers p.) [20]